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Carine Lucas, Christine Kazantsev

MULTISCALE APPROXIMATIONS FOR OCEAN EQUATIONS: THEORETICAL AND NUMERICAL RESULTS. *

CARINE LUCAS† AND CHRISTINE KAZANTSEV‡

Abstract. We study a stationary Quasi-Geostrophic type equation in one or two dimensional spaces, with a quickly varying topography. We consider an asymptotic expansion of this equation on several space and time scales. At each expansion's order, we split the approximated solution into an interior function, which represents the solution far from the western boundary, and a corrector function that takes into account the boundary layer. We derive the systems at each order for the two functions and prove mathematical properties on these systems. Then we present numerical tricks and results, with and without topography, in the one and two dimensional cases. The method is very efficient compared to classical ones (finite differences, finite elements) which are very expensive due to the quickly varying topography and thin boundary layer.

Key words. Multiscale, existence of solutions, numerical results, boundary layer.

subject classifications. 34E13, 76D10, 35A05, 65N06.

1. Introduction.

Introduced by Charney [4] in 1948 for the atmospheric motion, used in 1950 by Charney et al. [5] for the first computer weather forecasts on the ENIAC computer, the Quasi-Geostrophic potential vorticity equation (QG equation) is adapted to the motion of a wind-driven ocean by Bryan [2] in 1963. During the 70s, more realistic models are developed, like multilayer quasigeostrophic models (see [7]). Thanks to more powerful computers, primitive equations have replaced them for ocean forecasts. But QG models are still very useful simplified models, especially for preliminary studies.

In this paper, we are interested in the stationary QG equation in a domain with an ε-periodic bathymetry. Numerically, two problems occur: the first one is to resolve correctly the boundary layer, that implies to work with a rather small space scale near the western boundary. The second difficulty is due to the ε-periodic bathymetry, which imposes a very small space scale everywhere. The cost of usual methods (finite difference, finite elements) explodes. In this work, we follow the multiscale approach of R. Klein, E. Mikusky, A. Owinoh [9], P. Ailliot, E. Frénot, V. Monbet [1], D. Gérard-Varet [6] and D. Bresch, D. Gérard-Varet [3] and develop an asymptotic expansion of the QG equation. Writing the first order terms in ε, we get an approximate solution of this equation. But for each order, due to the presence of a boundary layer, we split the function in two parts: one deals with the interior of the domain, and the other one is a corrector which takes into account the western part of the domain. The systems obtained in this way are well-posed from a mathematical point of view, and easy to solve numerically. Notice that we only need to compute the first two orders of expansion to get a good approximation.

The paper is organized as follows: in Section 2, we start with a derivation of the systems based on a multiple scale approximation of the QG equation. We also give mathematical results for both the one dimensional and two dimensional QG-type

*Received date / Revised version date

†Laboratoire de Mathématiques et Applications, Physique Mathématique d'Orléans (MAPMO - UMR6628) Fédération Denis Poisson (FDP - FR2964) CNRS/Université d'Orléans, B.P. 6759, 45067 Orléans cedex 2, France (`Carine.Lucas@univ-orleans.fr`)

‡Université de Grenoble, INRIA and CNRS, Laboratoire Jean Kuntzmann, BP 53, 38041 Grenoble Cedex 9, France (`Christine.Kazantsev@imag.fr`).

equations. Section 3 presents the numerical method and results for the stationary equation in dimension one or two, without and with the ε-periodic topography. In Section 4, we come back to the one dimensional equation with topography. We show that the effects of the topography can be computed separately as a corrector-like function we must add to the solution of the equation with flat bottom. Consequently, the cost of the ε-periodic topography is low.

2. Theoretical developments.

In this part, we derive the systems based on a multiscale approximation of a Quasi-Geostrophic type equation. First, we present the equation itself and its main characteristics, and then we introduce the multiscale asymptotic developments, starting with a simple example in one dimension. In this case, we also prove some properties of well-posedness before giving the equations satisfied at each order. At this point, we are able to come back to the complete two dimensional stationary Quasi-Geostrophic equation and express the systems which give the first orders of the approximation of the solution.

2.1. The Quasi-Geostrophic equation.

The main goal of this paper is the study of the following stationary Quasi-Geostrophic type equation:

$$\begin{aligned} -\varepsilon^2\Delta^2\psi + J\left(\psi,\varepsilon^2\Delta\psi + b\left(\tfrac{x}{\varepsilon},y\right)+\varepsilon^{-1}y\right) &= \varepsilon^{-1}f(x,y) && \text{in } \mathcal{D}, \\ \psi &= 0 && \text{on } \partial\mathcal{D}, \\ \Delta\psi &= 0 && \text{on } \partial\mathcal{D}, \end{aligned} \tag{2.1}$$

where the unknown ψ is called the stream function, J is the jacobian defined by $J(\mathrm{f},\mathrm{g}) = \partial_x \mathrm{f}\partial_y \mathrm{g} - \partial_y \mathrm{f}\partial_x \mathrm{g}$, b is a periodic function, of null mean value (it represents the topography) and $\mathcal{D} = [0,1]\times[0,1]$. The parameter ε denotes a small number. We choose b such that, for x from 0 to 1, $b(x/\varepsilon)$ has an entire number of periods. The function f will vanish on the boundaries corresponding to $y=0$ and $y=1$ and is at the main order (in ε). We recall that the stream function ψ is given by $\psi = \nabla^\perp u$, where u is the velocity of the fluid.
Let us give a comment about this equation. When ε tends to zero, the type of the equation is modified: we do not need the boundary condition on the western part of the domain. So we have a boundary layer on the outflow boundary (for $x=0$).

2.2. The one dimensional case: mathematical results and multiscale system.

We start with the study of a restriction of Equation (2.1) to one dimension in space in order to introduce the method on a simple model (we add an evolution in time as it does not bring any real difficulty):

$$\begin{aligned} &\partial_t\psi(t,x) - \partial_x^2\psi(t,x) - \varepsilon^{-1}\partial_x\psi(t,x) = \varepsilon^{-1}f(t,x) && \text{in } [0,T]\times\mathcal{D}, \\ &\psi(t,0) = \psi(t,1) = 0 && \forall t\in[0,T], \\ &\psi(0,x) = 0 && \forall x\in\mathcal{D}, \end{aligned} \tag{2.2}$$

for $T\in\mathbb{R}_+^\star$, $\mathcal{D} =]0,1[$ and $f\in L^2\left(0,T;H^{-1}(\mathcal{D})\right)$. The parameter ε is supposed to be small.

2.2.1. Mathematical properties.

Let us give some mathematical properties of the solution of Equation (2.2). First, thanks to a classical analysis, if the source term f is in $L^2(0,T;H^{-1}(\mathcal{D}))$, the solution

ψ is in $L^\infty\left(0,T;L^2(\mathcal{D})\right)\cap L^2\left(0,T;H_0^1(\mathcal{D})\right)$.
We can also prove the following theorem using Lions theorem for parabolic equations (see [10]):

THEOREM 2.1. *We denote by* $(.,.)$ *the usual* $L^2(\mathcal{D})$ *scalar product and we define the bilinear form* a^ε *by*

$$a^\varepsilon(u(t),v)=\int_{\mathcal{D}}\partial_x u(t)\left(\partial_x v-\frac{1}{\varepsilon}v\right)dx.$$

Then, for all ε*, there exists a unique solution* $\psi^\varepsilon\in\mathcal{C}\left([0,T];L^2(\mathcal{D})\right)\cap L^2\left(0,T;H_0^1(\mathcal{D})\right)$ *such that, for all* $v\in H_0^1(\mathcal{D})$ *and all* $f\in L^2\left(0,T;H^{-1}(\mathcal{D})\right)$*,*

$$\frac{\partial}{\partial t}\left(\psi^\varepsilon(t,x),\mathrm{v}(x)\right)+a^\varepsilon\left(\psi^\varepsilon(t,x),\mathrm{v}(x)\right)=\frac{1}{\varepsilon}<f(t,x),\mathrm{v}(x)>_{H^{-1}(\mathcal{D})\times H_0^1(\mathcal{D})}$$

with $\psi^\varepsilon(0,x)=0$ *for all* $x\in\mathcal{D}$.

2.2.2. Approximate system based on a multiscale analysis.

In this part, we propose a new approximation method to solve Equation (2.2). We introduce a new space scale and we perform an asymptotic development, for a small ε. This gives an approximate solution of our problem.

Construction of the approximate solution.

We seek an approximate solution ψ_{app} of the problem (2.2) as the sum of an 'interior' term and a 'corrector' term that depends on the quick scale x/ε. We also decompose these two terms in powers of ε in order to write ψ_{app} as:

$$\psi_{app}(t,x)=\sum_{i=0}^{\infty}\varepsilon^i\left(\psi_i^{int}(t,x)+\psi_i^{cor}\left(t,\frac{x}{\varepsilon}\right)\right).$$

Let us suppose that the function f does not depend on the quick scale and write the equations satisfied by each ψ_i^{int} and ψ_i^{cor}.

The interior function, defined by $\psi_{app}^{interior}(t,x)=\sum_{i=0}^{\infty}\varepsilon^i\psi_i^{int}(t,x)$, is solution of the equation:

$$\partial_t\psi_{app}^{interior}(t,x)-\partial_x^2\psi_{app}^{interior}(t,x)-\frac{1}{\varepsilon}\partial_x\psi_{app}^{interior}(t,x)=\frac{1}{\varepsilon}f(t,x)\quad\text{in }[0,T]\times\mathcal{D},$$

which reads:

$$\sum_{i=0}^{\infty}\varepsilon^i\left(\partial_t\psi_i^{int}(t,x)-\partial_x^2\psi_i^{int}(t,x)-\frac{1}{\varepsilon}\partial_x\psi_i^{int}(t,x)\right)=\frac{1}{\varepsilon}f(t,x)\qquad\text{in }[0,T]\times\mathcal{D}.$$

We assume that f does not contain any term of order ε^j with $j>0$ and we identify the powers of ε; we find the equations:

$$\text{terms in }\tfrac{1}{\varepsilon}\quad\begin{cases}-\partial_x\psi_0^{int}(t,x)=f(t,x) & \text{in }[0,T]\times\mathcal{D},\\ \psi_0^{int}(t,1)=0 & \forall t\in[0,T],\end{cases}$$

$$\text{terms of higher order }(i\geq 1)\begin{cases}\partial_x\psi_i^{int}(t,x)=\partial_t\psi_{i-1}^{int}(t,x)-\partial_x^2\psi_{i-1}^{int}(t,x) & \text{in }[0,T]\times\mathcal{D},\\ \psi_i^{int}(t,1)=0 & \forall t\in[0,T].\end{cases}$$

We perform the same reasoning for the corrector term defined by the relation $\psi_{app}^{corrector}(t,y)=\sum_{i=0}^{\infty}\varepsilon^i\psi_i^{cor}(t,y)$ solution of the homogeneous equation. Here, the boundary conditions are such that the sum of the corrector and the interior terms satisfy Equation (2.2), that is:

$$\begin{cases}\psi_{app}^{corrector}(t,0)=-\psi_{app}^{interior}(t,0), & \forall t\in[0,T],\\ \psi_{app}^{corrector}\left(t,\dfrac{x}{\varepsilon}\right)\to 0 & \text{when } x \text{ is far from the boundary layer } (x\gg\varepsilon).\end{cases}$$

Remark 2.1. *These boundary conditions express the role of the corrector term. On the one hand, it must correct the interior solution such that their sum satisfies the conditions of the inital equation. On the other hand, far from the boundary layer, its influence should be very small.*

We define $X=x/\varepsilon$: for a fixed x, when $\varepsilon\to 0$, $X\to+\infty$. Then we can see the corrector $\psi_{app}^{corrector}$ as a function of X, which gives:

$$\begin{aligned}&\varepsilon^2\partial_t\psi_{app}^{corrector}(t,X)-\partial_X^2\psi_{app}^{corrector}(t,X)-\partial_X\psi_{app}^{corrector}(t,X)=0 && \text{in } [0,T]\times\mathbb{R}^+,\\ &\psi_{app}^{corrector}(t,0)=-\psi_{app}^{interior}(t,0) && \forall t\in[0,T],\\ &\lim_{X\to+\infty}\psi_{app}^{corrector}(t,X)=0.\end{aligned}$$

If we use the development of the corrector $\psi_{app}^{corrector}$, we get:

$$\begin{aligned}&\sum_{i=0}^{\infty}\varepsilon^i\left(\varepsilon^2\partial_t\psi_i^{cor}(t,X)-\partial_X^2\psi_i^{cor}(t,X)-\partial_X\psi_i^{cor}(t,X)\right)=0 && \text{in } [0,T]\times\mathbb{R}^+,\\ &\sum_{i=0}^{\infty}\varepsilon^i\psi_i^{cor}(t,0)=-\textstyle\sum_{i=0}^{\infty}\varepsilon^i\psi_i^{int}(t,0) && \forall t\in[0,T],\\ &\lim_{X\to+\infty}\sum_{i=0}^{\infty}\varepsilon^i\psi_i^{cor}(t,X)=0.\end{aligned}$$

As before, we identify the powers of ε to write a system for each ψ_i^{cor}:

$$\text{for } i=0 \text{ or } 1\quad\begin{cases}\partial_X^2\psi_i^{cor}(t,X)+\partial_X\psi_i^{cor}(t,X)=0 & \text{in } [0,T]\times\mathbb{R}^+,\\ \psi_i^{cor}(t,0)=-\psi_i^{int}(t,0) & \forall t\in[0,T],\\ \lim\limits_{X\to+\infty}\psi_i^{cor}(t,X)=0,\end{cases}$$

$$\text{and for } i\geq 2\quad\begin{cases}\partial_X^2\psi_i^{cor}(t,X)+\partial_X\psi_i^{cor}(t,X)=\partial_t\psi_{i-2}^{cor}(t,X) & \text{in } [0,T]\times\mathbb{R}^+,\\ \psi_i^{cor}(t,0)=-\psi_i^{int}(t,0) & \forall t\in[0,T],\\ \lim\limits_{X\to+\infty}\psi_i^{cor}(t,X)=0.\end{cases}$$

The condition $\lim_{X\to+\infty}\psi_i^{cor}(t,X)=0$ will be replaced by $\psi_i^{cor}(t,M)=0$, for M large enough.

Existence of the ψ_i^{int} and ψ_i^{cor}.

We start by studying the conditions for the existence of the interior functions. The function ψ_0^{int} is defined by $\partial_x\psi_0^{int}(t,x)=-f(t,x)$ with $\psi_0^{int}(t,1)=0$. If we want ψ_0^{int} in the space $L^\infty\left(0,T;L^2(\mathcal{D})\right)\cap L^2\left(0,T;H^1(\mathcal{D})\right)$, we have to take the source term f in $L^\infty\left(0,T;L^1(\mathcal{D})\right)\cap L^2\left(0,T;L^2(\mathcal{D})\right)$ and then ψ_0^{int} is in $L^\infty\left(0,T;\mathcal{C}(\bar{\mathcal{D}})\right)$. If we also

seek ψ_0^{int} as a continuous function in time, we must choose f in $\mathcal{C}\left([0,T];L^1(\mathcal{D})\right)$ and consequently ψ_0^{int} is in $\mathcal{C}([0,T];\bar{\mathcal{D}})$.
For the following orders, we easily prove by recurrence (see [10]) that it is necessary that f belongs to:

$$W^{i,\infty}\left(0,T;L^1(\mathcal{D})\right)\cap\cdots\cap W^{1,\infty}\left(0,T;W^{i-1,1}(\mathcal{D})\right)\cap L^\infty\left(0,T;W^{i,2}(\mathcal{D})\right)$$

to define ψ_i^{int} and if we want the continuity in time, we impose f to be in:

$$\mathcal{C}^i\left([0,T];L^1(\mathcal{D})\right)\cap\cdots\cap\mathcal{C}^1\left([0,T];W^{i-1,1}(\mathcal{D})\right)\cap\mathcal{C}\left([0,T];W^{i,2}(\mathcal{D})\right).$$

Looking carefully at the formulae for the corrector, we see that the existence of the ψ_i^{int} directly gives the existence of the ψ_i^{cor}.
Remark 2.2. *A priori we do not know that $\psi_{app}(0,x)=0$, for all x in $\mathcal{D}$, that is: $\sum_{i=0}^{\infty}\varepsilon^i\left(\psi_i^{int}(0,x)+\psi_i^{cor}\left(0,\frac{x}{\varepsilon}\right)\right)=0,\ \forall x\in\mathcal{D}$.*
So we impose $f(0,x)=0$, for all x in $\mathcal{D}$ that ensures that the boundary conditions of (2.2) are satisfied when ε tends to zero.

Convergence when ε tends to zero.

We choose an integer $N\geq 0$ and, for f regular enough, we define the approximation at the order N by:

$$\tilde{\psi}_{app}^N(t,x)=\sum_{i=0}^{N}\varepsilon^i\left(\psi_i^{int}(t,x)+\psi_i^{cor}\left(t,\frac{x}{\varepsilon}\right)\right).$$

With classical methods (see [10]), we prove that, when ε tends to zero, the approximate solution at the order N $\tilde{\psi}_{app}^N$ converges to the solution ψ of Equation (2.2).

2.3. Systems in two dimensions.

We now apply the previous method to the two dimensional case (2.1). The quick scale X is still defined by $X=x/\varepsilon$.

If we perform a classical asymptotic development in ψ, at the order $1/\varepsilon^2$, we get the following equation:

$$\begin{cases}-\partial_X^4\psi_0(x,X,y)+\partial_X\psi_0(x,X,y)=0 & \text{in } \mathcal{D},\\ \partial_X^2\psi_0(x,X,y)=0 & \text{for } x=0 \text{ and } x=1,\\ \psi_0(x,X,y)=0 & \text{for } x=0 \text{ and } x=1.\end{cases}$$

Then we see that, as in one dimension, we can split ψ_0 into an 'interior' term $\psi_0^{int}(x,X,y)$ and a 'corrector' term $\psi_0^{cor}(X,y)$. The 'interior' term $\psi_0^{int}(x,X,y)$, that depends on the X variable due to the X dependence of the topography function b, must vanish for $x=1$ and is periodic in X. The 'corrector' term $\psi_0^{cor}(X,y)$ has an influence only on the boundary layer (it vanishes far from the western coast) and must be equal to the opposite of ψ_0^{int} on the western boundary of the domain.
Now we are able to write the two systems for the two terms at the first order:

$$\begin{cases}-\partial_X^4\psi_0^{int}(x,X,y)+\partial_X\psi_0^{int}(x,X,y)=0 & \text{in } \mathcal{D},\\ \partial_X^2\psi_0^{int}(0,0,y)=0 \text{ and } \partial_X^2\psi_0^{int}(1,\frac{1}{\varepsilon},y)=0 & \text{for } y\in[0,1],\\ \psi_0^{int}(1,y)=0 & \text{for } y\in[0,1],\end{cases}$$

which means that ψ_0^{int} does not depend on X (due to the periodicity). Moreover, for the corrector, we have:

$$\boxed{\begin{cases} -\partial_X^4 \psi_0^{cor}(X,y) + \partial_X \psi_0^{cor}(X,y) = 0 & \text{in } \mathcal{D}, \\ \partial_X^2 \psi_0^{cor}(X,y) = 0 & \text{for } X=0 \text{ and } X = \frac{1}{\varepsilon}, \\ \psi_0^{cor}(0,y) = -\psi_0^{int}(0,y), & \text{for } y \in [0,1] \end{cases}} \tag{2.3}$$

and can be computed as soon as we know the interior term. Note that the dependence in the y variable is only related to $\psi_0^{int}(0,y)$ and can be added afterwards.

Up to now, we have written the first order term of Equation (2.1). We have expressed the corrector ψ_0^{cor} as a function of ψ_0^{int} and we know that ψ_0^{int} does not depend on X. But we do not have any equation for ψ_0^{int}. So we are led to study the next order.

Equation (2.1) at the order $1/\varepsilon$ gives:

$$\begin{cases} -\partial_X^4 \psi_1 - 4\partial_X^3 \partial_x \psi_0 + \partial_X \psi_0 \partial_y \partial_X^2 \psi_0 + \partial_x \psi_0 & \\ \qquad + \partial_X \psi_1 - \partial_y \psi_0 \partial_X^3 \psi_0 - \partial_y \psi_0 \partial_X b = f & \text{in } \mathcal{D}, \\ \partial_X^2 \psi_1 = 0 & \text{for } x=0 \text{ and } x=1, \\ \psi_1 = 0 & \text{for } x=0 \text{ and } x=1. \end{cases}$$

Like before, we split the first order ψ_1 into ψ_1^{int}, such that ψ_1^{int} is periodic in X, and ψ_1^{cor}, equal to zero almost everywhere except near the boundary $\{0\} \times [0,1]$. We have the equality:

$$-\partial_X^4 \psi_1^{int} + \partial_x \psi_0^{int} + \partial_X \psi_1^{int} - \partial_y \psi_0^{int} \partial_X b = f.$$

We compute the mean value in X, using the periodicity and the fact that ψ_0^{int} does not depend on X, and we get $\partial_x \psi_0^{int}(x,y) = f(x,y)$, so we have an expression of ψ_0^{int} on $\mathcal{D}$:

$$\boxed{\psi_0^{int}(x,y) = \int_1^x f(s,y)\, ds.}$$

Moreover, we have information on ψ_1^{int}: the X variation is given by:

$$\boxed{-\partial_X^4 \psi_1^{int} + \partial_X \psi_1^{int} = \partial_y \psi_0^{int} \partial_X b \quad \text{in } \mathcal{D},}$$

with $\partial_X^2 \psi_1^{int}$ equal to zero on the left and right boundaries of the domain $\mathcal{D}$, and ψ_1^{int} vanishes for $X = 1/\varepsilon$.

Then, if we want to know ψ_1^{int}, we are led to identify the dependence of ψ_1^{int} in x, that is the function $D(x,y)$ such that:

$$\boxed{\psi_1^{int}(x,X,y) = \int_0^X \partial_{\widetilde{X}} \psi_1^{int}(x,\widetilde{X},y)\, d\widetilde{X} + D(x,y),}$$

and with $D(1,y)$ given so that ψ_1^{int} vanishes on the right boundary of the domain. We will compute D thanks to Equation (2.1) at the order 1.

Before going to the next order, we can also express ψ_1^{cor}; it must vanish far from the boundary layer and satisfy:

$$\begin{cases} -\partial_X^4 \psi_1^{cor} + \partial_X \psi_1^{cor} = -\partial_X \psi_0^{cor} \partial_y \partial_X^2 \psi_0^{cor} \\ \quad + \partial_y \psi_0^{cor} \partial_X^3 \psi_0^{cor} + \partial_y \psi_0^{int} \partial_X^3 \psi_0^{cor} + \partial_y \psi_0^{cor} \partial_X b & \text{in } \mathcal{D}, \\ \partial_X^2 \psi_1^{cor} = 0 & \text{for } X=0 \text{ and } X = \frac{1}{\varepsilon}, \\ \psi_1^{cor}(0,0,y) = -\psi_1^{int}(0,0,y) & \text{for } y \in [0,1]. \end{cases}$$

We just need the first order and ψ_1^{int} to get ψ_1^{cor}.

So we have to express the function D to have the complete first order solution. To do so, we write Equation (2.1) at the order 1:

$$\begin{cases} -\partial_X^4 \psi_2 - 4\partial_X^3 \partial_x \psi_1 - 6\partial_X^2 \partial_x^2 \psi_0 - 2\partial_X^2 \partial_y \psi_0 + \partial_x \psi_0 \partial_y \partial_X^2 \psi_0 + \partial_X \psi_1 \partial_y \partial_X^2 \psi_0 \\ \quad + 2\partial_X \psi_0 \partial_y \partial_x \partial_X \psi_0 + \partial_X \psi_0 \partial_y \partial_X^2 \psi_1 + \partial_x \psi_0 \partial_y b + \partial_X \psi_1 \partial_y b + \partial_x \psi_1 \\ \quad + \partial_X \psi_2 - 3\partial_y \psi_0 \partial_x \partial_X^2 \psi_0 - \partial_y \psi_1 \partial_X^3 \psi_0 - \partial_y \psi_0 \partial_X^3 \psi_1 - \partial_y \psi_1 \partial_X b = 0 & \text{in } \mathcal{D}, \\ \partial_X^2 \psi_2 = -\partial_x^2 \psi_0 - \partial_y^2 \psi_0 & \text{for } x=0 \text{ and } x=1, \\ \psi_2 = 0 & \text{for } x=0 \text{ and } x=1. \end{cases}$$

Again, we split into interior and corrector functions and we study the former:

$$\begin{aligned} -\partial_X^4 \psi_2^{int} - 4\partial_X^3 \partial_x \psi_1^{int} + \partial_x \psi_0^{int} \partial_y b + \partial_X \psi_1^{int} \partial_y b + \partial_x \psi_1^{int} + \partial_X \psi_2^{int} \\ - \partial_y \psi_0^{int} \partial_X^3 \psi_1^{int} - \partial_y \psi_1^{int} \partial_X b = 0, \end{aligned}$$

and we compute the mean value in X, with ψ_2^{int} periodic as for the previous orders. Then we have:

$$\begin{aligned} -\overline{4\partial_X^3 \partial_x \psi_1^{int}}^X + \partial_x \psi_0^{int} \overline{\partial_y b}^X + \overline{\partial_X \psi_1^{int} \partial_y b}^X + \overline{\partial_x \psi_1^{int}}^X \\ - \partial_y \psi_0^{int} \overline{\partial_X^3 \psi_1^{int}}^X - \overline{\partial_y \psi_1^{int} \partial_X b}^X = 0, \end{aligned}$$

which gives $\partial_x D(x,y)$. Since $D(1,y) = 0$ for all y in $[0,1]$, a simple integration from $x=1$ gives D (we give its explicit formulation in the following, in some particular cases).

Remark 2.3. *Note that it does not ensure that ψ vanishes on the boundaries $[0,1] \times \{0\}$ and $[0,1] \times \{1\}$. In the following, we prove that, if we consider a topography that vanishes on these boundaries, this property is satisfied.*

In the same way, Equation (2.1) at the order 1 gives the variation of ψ_2^{int} in X. The dependence in x will be expressed thanks to the average in X of Equation (2.1) at the order ε. For the corrector, we just need the previous terms.

Then, writing Equation (2.1) at each order, we can obtain each term of the asymptotic development.

2.3.1. A particular case: b does not depend on y.

We suppose that b does not depend on y and we study the first order interior equations. First, we define a function $G(X)$ solution of:

$$\begin{cases} -G^{(4)}(X) + G'(X) = b'(X) & \text{in } [0, 1/\varepsilon], \\ G''(X) = 0 & \text{for } X=0 \text{ and } X = 1/\varepsilon, \\ G(X) = 0 & \text{for } X=0 \text{ and } X = 1/\varepsilon. \end{cases}$$

Then ψ_1^{int} reads $\psi_1^{int}(x,X,y) = \partial_y\psi_0^{int}(x,y)\,G(X) + D(x,y)$ with the following equation for D:

$$\begin{cases} \partial_x D(x,y) + \partial_x\partial_y\psi_0^{int}(x,y)\overline{G(X)}^X - \partial_y^2\psi_0^{int}(x,y)\overline{G(X)\partial_X b}^X = 0 & \text{in } \mathcal{D}, \\ D(1,y) = 0 & \text{for } y \in [0,1]. \end{cases}$$

This system can be easily solved numerically.

2.3.2. Another particular case: b with separable variables.

We choose b as $b(X,y) = b_1(X)b_2(y)$, with b_1 periodic and with a null mean value. As before, we define a function $\widetilde{G}$ solution of:

$$\begin{cases} -\widetilde{G}^{(4)}(X) + \widetilde{G}'(X) = b_1'(X) & \text{in } [0,1/\varepsilon], \\ \widetilde{G}''(X) = 0 & \text{for } X = 0 \text{ and } X = 1/\varepsilon, \\ \widetilde{G}(X) = 0 & \text{for } X = 0 \text{ and } X = 1/\varepsilon, \end{cases}$$

and ψ_1^{int} is given by $\psi_1^{int}(x,X,y) = \partial_y\psi_0^{int}(x,y)\,b_2(y)\,\widetilde{G}(X) + \widetilde{D}(x,y)$. The equation to solve to get $\widetilde{D}$ is much more intricate than in the previous case:

$$\begin{cases} \partial_y\psi_0^{int}(x,y)\,b_2(y)\,b_2'(y)\overline{b_1(X)\widetilde{G}'(X)}^X + \partial_x\widetilde{D}(x,y) + \partial_x\partial_y\psi_0^{int}(x,y)\,b_2(y)\,\overline{\widetilde{G}(X)}^X \\ \quad -b_2(y)\left(\partial_y^2\psi_0^{int}(x,y)\,b_2(y) + \partial_y\psi_0^{int}(x,y)\,b_2'(y)\right)\overline{\widetilde{G}(X)\,b_1'(X)}^X = 0 & \text{in } \mathcal{D}, \\ \widetilde{D}(1,y) = 0 & \text{for } y \in [0,1]. \end{cases}$$

These relations can also be solved without any real difficulty.

3. Numerical results.

In this part, we give numerical results obtained thanks to the previous derivations. We study the first and the second orders of both the interior and corrector functions, in the one and two dimensional cases. We also explain how we changed the mathematical equalities into relations that can be implemented easily.

3.1. One dimensional case.

3.1.1. Implementation.

The system have been implemented using Fortran. We define two different space steps (in the boundary layer and outside the boundary layer). We choose centered finite differences to compute the ψ_i^{cor} and the ψ_i^{int}, and for the former we use the LU decomposition with derivatives at the second order. We also approximate the solution of Equation (2.2) by centered finite differences and LU decomposition but without asymptotic development.

3.1.2. Stationary results.

We consider the case $f(t,x) = x$ in order to be able to express the exact solution. One can also note that, in this case, the terms in the asymptotic development ψ_i vanish from $i = 2$.

We find that the approximated solution is very close to the exact solution and, contrarily to the "usual" solution, we do not have a strong stability condition and we are less restricted to choose the parameters.

3.1.3. Improvements for the evolution equation.

Then, we introduce the evolution in time (we just have to add some time derivatives). We need to reduce the computation time of the corrector: the condition $\lim_{X\to+\infty}\psi_i^{cor}(t,X)=0$ is replaced by $\psi_i^{cor}(t,M)=0$ for M large enough. The question is: how to choose M ?
The first idea is to take M such that εM is outside of the domain. If we look at the shape of the corrector, we see that its contribution is significant only on a small part of the domain, and more precisely only on the boundary layer. Unfortunately, the computation time is not significantly reduced.
Consequently, another idea is to compute the corrector only in the boundary layer. This method will be validated and used in what follows.

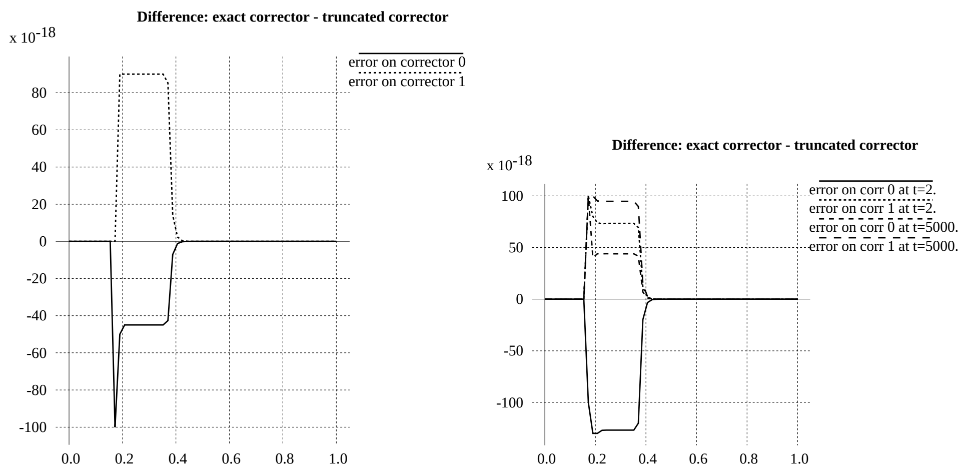

FIG. 3.1. *Difference between the correctors for $f(t,x)=x$ (on the left) and $f(t,x)=x+sin(t)$ (on the right), for $\varepsilon=0.01$ and a time step equal to 0.1. We define the boundary layer length as 0.1, and consider 31 points in the boundary layer, 51 points outside.*

In order to prove the accuracy of the reduction of the support length, we compare the exact corrector and the corrector that vanishes after a bound, chosen as one and a half times the boundary layer size. The results, presented in Figure 3.1, prove that, for both the stationary and the evolution problems, the truncated corrector is close to the exact one, even for long times, but regarding the computation time, our approximation is much better.

Then, we have an efficient method to compute the solution of Equation (2.2), based on asymptotic developments and a truncation of the boundary layer corrector term. In the following, we consider the two dimensional case and give some numerical results.

3.2. Numerical results for the 2D case.

3.2.1. Programming issues.

The first difficulty due to this new equation is the complexity of the equation for the corrector, which is not of order two any more but of order 4. As noted above, we will add the y variation afterwards (we substitute $\psi_0^{cor}(0,y)$ with 1). We use a centered scheme to express the fourth derivative. To get the second derivative on the

boundaries, we introduce a fictive point on the exterior of the domain, which can be combined with the relation for the fourth derivative.
The next problem occurs the computation of ψ_1^{cor}. We have an equation in the X variable whereas ψ_0^{int} depends on x and y and we recall that $X = x/\varepsilon$. We consider ψ_1^{int} as a piecewise constant function in X, i.e. for $\varepsilon X \in [x_i, x_{i+1}]$, $\psi_1^{int}(\varepsilon X, y) = \psi_1^{int}(x_i, y)$.

3.2.2. Numerical results.
We take $f(x,y) = -\sin(2\pi y)$ on a 500×500 grid.

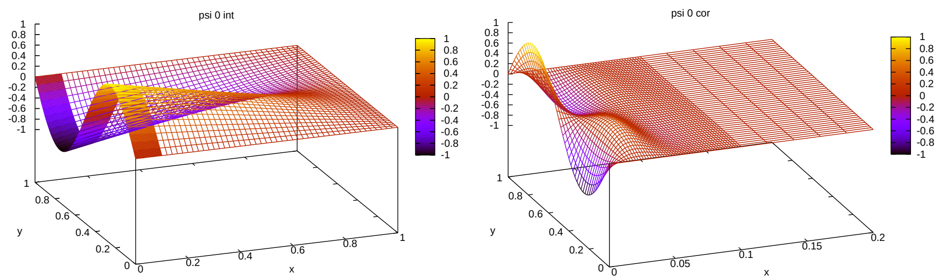

FIG. 3.2. *Plots of* ψ_0^{int} *and* ψ_0^{cor} *for* $\varepsilon = 0.01$.

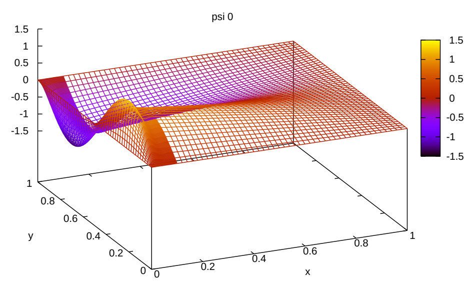

FIG. 3.3. *Plot of* $\psi_0 = \psi_0^{int} + \psi_0^{cor}$ *for* $\varepsilon = 0.01$.

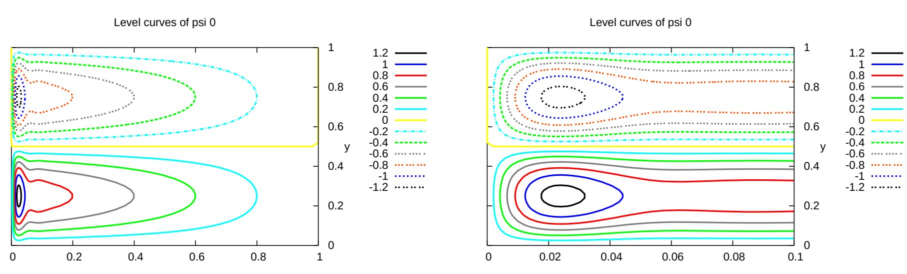

FIG. 3.4. *Level curves of* $\psi_0 = \psi_0^{int} + \psi_0^{cor}$ *for* $\varepsilon = 0.01$.

At the first order, the topography does not play any role, and, as for the one dimensional case, the corrector can be considered as equal to zero beyond a given bound (we chose here $x = 0.2$, see Figures 3.2 to 3.4).

In Figures 3.6 to 3.8, we chose the bottom profile function $b(X,y)=1/2\cos(\pi X/10)$. Here again, we can remark that the bottom has a non-negligible influence over the whole domain but also in the boundary layer.

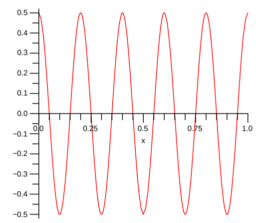

FIG. 3.5. *Topography:* $b(X)=\frac{1}{2}\cos\left(\frac{\pi X}{10}\right)$.

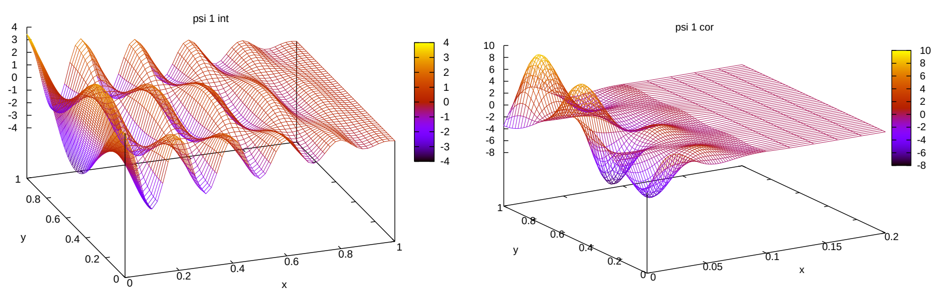

FIG. 3.6. *Plots of* ψ_1^{int} *and* ψ_1^{cor} *for* $\varepsilon=0.01$.

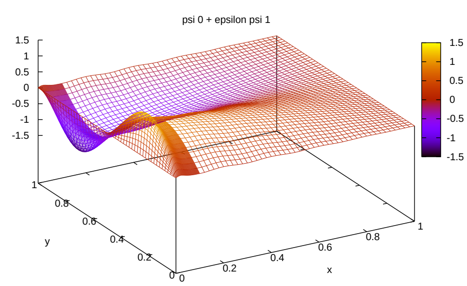

FIG. 3.7. *Plot of* $\psi_0+\varepsilon\psi_1=\psi_0^{int}+\psi_0^{cor}+\varepsilon\psi_1^{int}+\varepsilon\psi_1^{cor}$ *for* $\varepsilon=0.01$.

We have also tested the topography given by $b(X,y)=1/2\cos(\pi X/10)\sin(\pi y)$ and the results are plotted in Figures 3.10 to 3.12. As the topography vanishes on the boundaries $y=0$ and $y=1$, the boundary condition is satisfied by the sum $\psi_0+\varepsilon\psi_1$.

Now we have a program that can compute quickly the solution of Equation (2.1), even for small ε (such as 0.001). However, we cannot assert that the boundary conditions are satisfied for $y=0$ or 1. In the following section, we compare our results to the classical Quasi-Geostrophic model.

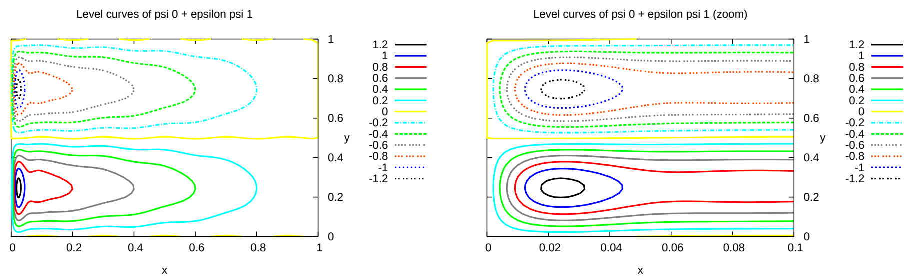

FIG. 3.8. *Level curves of* $\psi_0+\varepsilon\psi_1=\psi_0^{int}+\psi_0^{cor}+\varepsilon\psi_1^{int}+\varepsilon\psi_1^{cor}$ *for* $\varepsilon=0.01$.

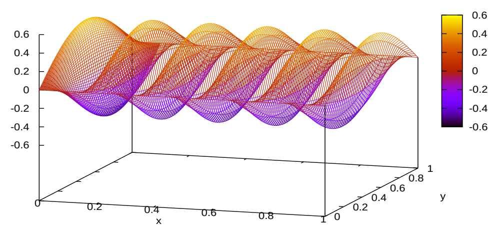

FIG. 3.9. *Topography:* $b(X,y)=\frac{1}{2}\cos\left(\frac{\pi X}{10}\right)\sin(\pi y)$.

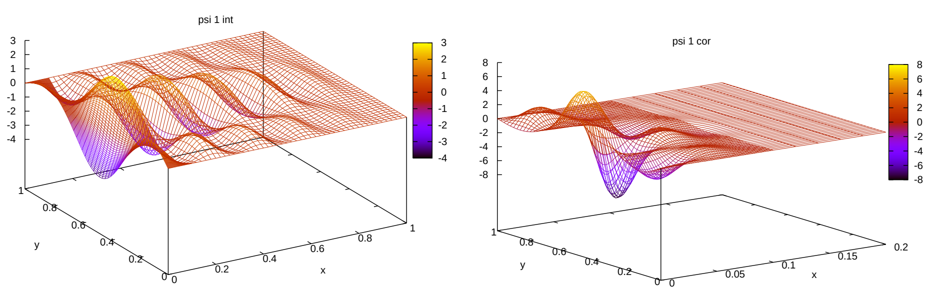

FIG. 3.10. *Plots of* ψ_1^{int} *and* ψ_1^{cor} *for* $\varepsilon=0.01$.

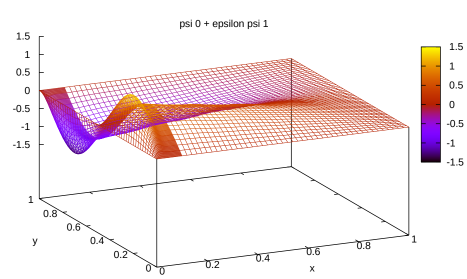

FIG. 3.11. *Plot of* $\psi_0+\varepsilon\psi_1=\psi_0^{int}+\psi_0^{cor}+\varepsilon\psi_1^{int}+\varepsilon\psi_1^{cor}$ *for* $\varepsilon=0.01$.

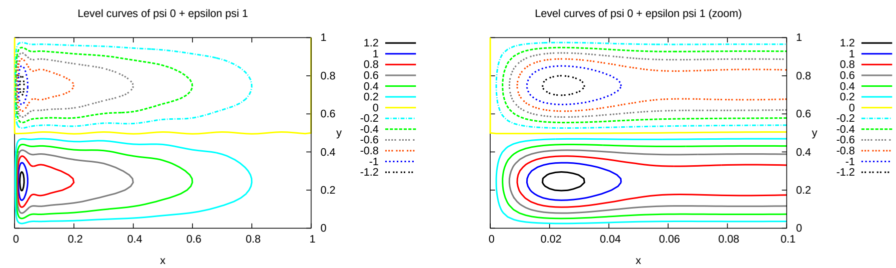

FIG. 3.12. *Level curves of* $\psi_0+\varepsilon\psi_1=\psi_0^{int}+\psi_0^{cor}+\varepsilon\psi_1^{int}+\varepsilon\psi_1^{cor}$ *for* $\varepsilon=0.01$.

3.2.3. Comparison.

In order to compare our model to a classical Quasi-Geostrophic model, we consider the equation:

$$\partial_t\Delta\psi+\sigma\Delta\psi-\Delta^2\psi+J\left(\psi,\Delta\psi+\mathrm{b}\left(\frac{x}{\varepsilon},y\right)+\varepsilon^{-3}y\right)=\mathrm{f}(x,y). \tag{3.1}$$

Note that multiplying Equation (3.1) by ε^2 and assuming $\sigma=0$, $\mathrm{b}\left(\frac{x}{\varepsilon},y\right)=\varepsilon^{-2}b\left(\frac{x}{\varepsilon},y\right)$ and $\mathrm{f}(x,y)=\varepsilon^{-3}f(x,y)$, the stationary regime satisfies Equation (2.1).

In Figure 3.13, we present the differences between the two programs, discretized with 200 points in both cases.

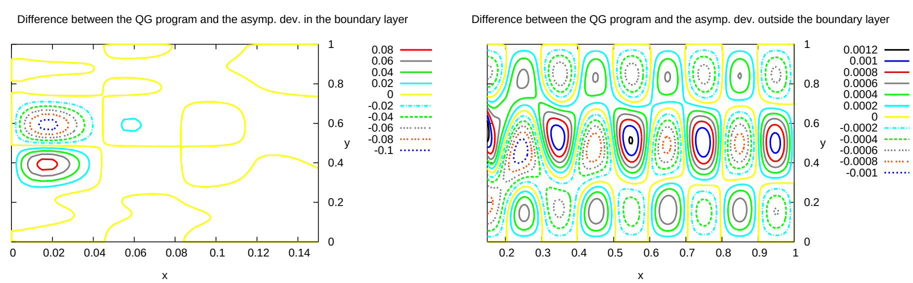

FIG. 3.13. *Difference between the two models: on the left, on the boundary layer and on the right, outside the boundary layer.*

We study separately the boundary layer. Outwards, the difference is very small, of order of 0.12%, see Figure 3.13, right. As ε is equal to 10^{-2}, we have a good approximation.

However, in the boundary layer, the approximation is not so accurate: let us study this phenomenon. The width of the boundary layer, following [8], is $2\pi\varepsilon/\sqrt{3}\approx 0.036$ which means that the classical program have only 7 points in this region, which is not well solved. Our approximation gives a better result thanks to the new variable X: the computation uses one hundred times more points in the boundary layer.

We can also prove that our approximation is at the second order: for large enough ε (of order of 0.1 to have a good classical solution) we plot the infinite norm of the difference between the two results. In Figure 3.14, which represents the error as a function of ε in a log-log scale, we get a line with a slope equal to 2, so the error is in ε^2.

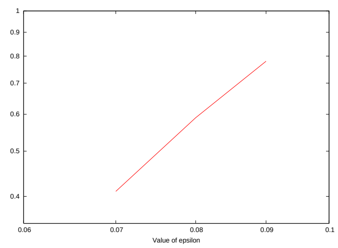

FIG. 3.14. *Difference between the two programs in a log-log scale.*

4. Back to the one dimensional case: study of the topography.

We can add the topography in Equation (2.2) and consider the following problem:

$$\begin{aligned} &\partial_t \psi(t,x) - \partial_x^2 \psi(t,x) + b\left(\frac{x}{\varepsilon}\right)\partial_x \psi(t,x) - \frac{1}{\varepsilon}\partial_x \psi(t,x) = \frac{1}{\varepsilon} f(t,x) && \text{in } [0,T]\times\mathcal{D}, \\ &\psi(t,0) = \psi(t,1) = 0 && \forall t \in [0,T], \\ &\psi(0,x) = 0 && \forall x \in \mathcal{D}, \end{aligned} \tag{4.1}$$

with $\mathcal{D} =]0,1[$, $T \in \mathbb{R}_+^\star$, $f \in L^2\left(0,T;H^{-1}(\mathcal{D})\right)$. The function b is periodic, its period is P, its mean value is zero, and, for x from 0 to 1, $b(x/\varepsilon)$ has an entire number of periods. Note that we could assert the same *a priori* estimates and existence results as in Section 2.2.

We are interested in the role of the topography in this equation: we write the asymptotic development as before and we isolate the bottom effect.

4.1. Asymptotic development.

As for a flat bottom, we split the unknown function into interior terms, that are periodic in x/ε, and corrector terms. The time-dependence does not really change the problem, so, for the sake of simplicity, we consider the stationary equation. Thanks to the periodicity, we get a system for the interior function. We will write it at each order in ε:

$$\begin{cases} -\partial_x^2 \psi^{interior}\left(x, \dfrac{x}{\varepsilon}\right) + b\left(\dfrac{x}{\varepsilon}\right)\partial_x \psi^{interior}\left(x, \dfrac{x}{\varepsilon}\right) - \dfrac{1}{\varepsilon}\partial_x \psi^{interior}\left(x, \dfrac{x}{\varepsilon}\right) = \dfrac{1}{\varepsilon} f(x), \\ \psi^{interior}(x=1) = 0. \end{cases} \tag{4.2}$$

The non-periodic part gives the system for the boundary layer corrector:

$$\begin{cases} -\partial_x^2 \psi^{corrector}\left(\dfrac{x}{\varepsilon}\right) + b\left(\dfrac{x}{\varepsilon}\right)\partial_x \psi^{corrector}\left(\dfrac{x}{\varepsilon}\right) - \dfrac{1}{\varepsilon}\partial_x \psi^{corrector}\left(\dfrac{x}{\varepsilon}\right) = 0, \\ \psi^{corrector}(X=0) = -\psi^{interior}(x=0), \\ \psi^{corrector}(X) \underset{X \to +\infty}{\longrightarrow} 0. \end{cases} \tag{4.3}$$

Let us study separately these two systems at the first orders in ε.

4.2. The interior function.

The function $\psi^{interior}$ is periodic in x/ε and should satisfy Equation (4.2). We still denote by X the quick scale x/ε, and we perform an asymptotic development of our function, writing $\psi^{interior}$ as the sum of the ψ_i^{int}.

At the order $1/\varepsilon^2$, we have $-\partial_X^2\psi_0^{int}(x,X)-\partial_X\psi_0^{int}(x,X)=0$. The function ψ_0 is assumed to be periodic in the quick scale, so it does not depend on X and we have to study the next order to have more information.

At the order $1/\varepsilon$, Equation (4.2) reads:

$$-\partial_X^2\psi_1^{int}(x,X)-\partial_x\psi_0^{int}(x)-\partial_X\psi_1^{int}(x,X)=f(x). \tag{4.4}$$

We integrate in X and, with the periodicity, we find: $\psi_0^{int}(x)=\displaystyle\int_x^1 f(s)\,ds$.

Moreover, when we replace this value in Equation (4.4), we obtain (exactly as for ψ_0^{int}) that ψ_1^{int} does not depend on X.

Since the functions ψ_0^{int} and ψ_1^{int} do not depend on X, Equation (4.2) at the order 1 gives:

$$-\partial_x^2\psi_0^{int}(x)-\partial_X^2\psi_2^{int}(x,X)+b(X)\partial_x\psi_0^{int}(x)-\partial_x\psi_1^{int}(x)-\partial_X\psi_2^{int}(x,X)=0.$$

With the mean value in X, we find $\psi_1^{int}(x)=f(x)-f(1)$ and we have the X derivative of ψ_2^{int}:

$$\partial_X\psi_2^{int}(x,X)=\partial_x\psi_0^{int}(x)\left[\int_0^X b(s)e^s\,ds+\frac{1}{e^P-1}\int_0^P b(s)e^s\,ds\right]e^{-X}.$$

Then we know ψ_2^{int} up to a function $\boldsymbol{D}(x)$ that can be determined with the following order.

At the order ε, Equation (4.2) becomes:

$$\begin{aligned}-\partial_x^2\psi_1^{int}(x)-2\partial_x\partial_X\psi_2^{int}(x,X)-\partial_X^2\psi_3^{int}(x,X)+b(X)\partial_x\psi_1^{int}(x)\\+b(X)\partial_X\psi_2^{int}(x,X)-\partial_x\psi_2^{int}(x,X)-\partial_X\psi_3^{int}(x,X)=0.\end{aligned}$$

Its mean value can be written as

$$-\partial_x^2\psi_1^{int}(x)+\overline{b(X)\partial_X\psi_2^{int}(x,X)}^X-\overline{\partial_x\psi_2^{int}(x,X)}^X=0,$$

where $\overline{\mathrm{g}}^X$ is the mean value of the function g in X.
Replacing the expression of $\partial_X\psi_2^{int}$ in this equality, we obtain:

$$-\partial_x\boldsymbol{D}(x)=N\partial_x^2\psi_0^{int}(x)-M\partial_x\psi_0^{int}(x)+\partial_x^2\psi_1^{int}(x),$$

where M and N depend only on b:

$$M=\overline{b(X)\left[\int_0^X b(s)e^s\,ds+\frac{1}{e^P-1}\int_0^P b(s)e^s\,ds\right]e^{-X}}^X,$$

$$N=\overline{\int_0^X\left(\int_0^u b(s)e^s\,ds+\frac{1}{e^P-1}\int_0^P b(s)e^s\,ds\right)e^{-u}\,du}^X.$$

Then ψ_2^{int} is completely determined by the expression:

$$\psi_2^{int}(x,X)=\partial_x\psi_0^{int}(x)\int_0^X\left(\int_0^u b(s)e^s\,ds+\frac{1}{e^P-1}\int_0^P b(s)e^s\,ds\right)e^{-u}\,du+\boldsymbol{D}(x).$$

We could carry on this process to get ψ_3^{int} and more generally each ψ_i^{int} but we do not give the details here.

Remark 4.1. *If we add the time dependence, ψ_1^{int} is solution of:*

$$\partial_x \psi_1^{int}(t,x) = -\partial_x^2 \psi_0^{int}(t,x) + \partial_t \psi_0^{int}(t,x)$$

and the equation for $\boldsymbol{D}$ becomes

$$-\partial_x \boldsymbol{D}(t,x) = N\partial_x^2 \psi_0^{int}(t,x) - M\partial_x \psi_0^{int}(t,x) + \partial_x^2 \psi_1^{int}(t,x) - \partial_t^2 \psi_1^{int}(t,x).$$

We have proved that we could give the expression of all the interior terms; let us consider now the boundary layer corrector terms.

4.3. Corrector terms for the boundary layer.

We develop the function $\psi^{corrector}$ as the sum of the ψ_i^{cor} and we isolate each order of Equation (4.3).

At the order 1, the equation and the boundary conditions give the expression of ψ_0^{cor} directly: $\psi_0^{cor}(X) = -\left(\int_0^1 f(s)\,ds\right)\exp(-X)$.

Equation (4.3) at the order ε reads:

$$\partial_X^2 \psi_1^{cor}(X) + \partial_X \psi_1^{cor}(X) = b(X)\partial_X \psi_0^{cor}(X),$$

$$\psi_1^{cor}(0) = -\psi_1^{int}(0),$$

$$\psi_1^{cor}(X) \underset{X\to+\infty}{\longrightarrow} 0.$$

Consequently, ψ_1^{cor} is completely determined by the value of ψ_1^{int} for $x=0$.

The next orders lead us to the same conclusion: if we know all the $(\psi_i^{cor})_{i\le j}$ and $(\psi_i^{int})_{i\le j+1}$, then we can compute ψ_{j+1}^{cor}.

Remark 4.2. *If we consider the evolution equation, from the third order, we must add the term $\partial_t \psi_{i-2}^{cor}$ in the right hand side of the equation for ψ_i^{cor}.*

We are now able to compute the first terms of the asymptotic development of the stationary solution of Equation (4.1) with the topography. In the following, we identify the topography effect through these equations and we show how to add this effect to the solution for a flat bottom.

4.4. Topography effect.

In this part, we denote by ψ the solution for a flat bottom, as in section 2.2.2, and by a tilde the solution of Equation (4.1) that, as in the previous sections, depends on an oscillating bottom.

Our goal is to express the effect of the topography, that is the term we could add to ψ to get $\widetilde{\psi}$. To do so, we compare the first orders (we stop at the third one) of the solutions of the equations with and without topography, and we find the equation of a so-called 'topography effect'.

We remark that the two developments are equal at the first order, and at the second order for the interior. So we focus on the differences between $\widetilde{\psi}_i^{int}$ and ψ_i^{int} for $i\ge 2$ and between $\widetilde{\psi}_i^{cor}$ and ψ_i^{cor} for $i\ge 1$.

4.4.1. Topography effect on the boundary layer at the order 1.

As we have previously mentioned, at the order 1 the two terms differ in the boundary layer only. If we denote by ϕ_1 this difference, that is $\phi_1(t,X)=\widetilde{\psi}_1^{cor}(t,X)-\psi_1^{cor}(t,X)$, it must satisfy the following equation:

$$\begin{aligned}
&\partial_X^2\phi_1(t,X)+\partial_X\phi_1(t,X)=b(X)\partial_X\widetilde{\psi}_0^{cor}(t,X)=b(X)\left(\int_0^1 f(t,s)\,ds\right)\exp(-X),\\
&\phi_1(t,X=0)=0,\\
&\phi_1(t,X)\underset{X\to+\infty}{\longrightarrow}0.
\end{aligned}$$

So if we want an approximation of the solution with topography $\widetilde{\psi}$ at the order 1, we just have to add the function ϕ_1 to the solution ψ for a flat bottom.

4.4.2. Topography effect at the order 2.

At the order 2, we have to consider the corrector terms but also the interior terms, as they both differ.

In the boundary layer, we define $\phi_{2CL}(t,X)=\widetilde{\psi}_2^{cor}(t,X)-\psi_2^{cor}(t,X)$ the solution of the system

$$\begin{aligned}
&\partial_X^2\phi_{2CL}(t,X)+\partial_X\phi_{2CL}(t,X)=b(X)\partial_X\widetilde{\psi}_1^{cor}(t,X),\\
&\phi_{2CL}(t,X=0)=\widetilde{\psi}_2^{cor}(t,X=0)-\psi_2^{cor}(t,X=0),\\
&\phi_{2CL}(t,X)\underset{X\to+\infty}{\longrightarrow}0.
\end{aligned}$$

Since we are able to replace $\partial_X\widetilde{\psi}_1^{cor}(t,X)$ by its value, namely:

$$e^{-X}\int_0^1 f(t,s)\,ds\left[\int_0^X b(s)\,ds-\int_0^{+\infty}e^{-y}\int_0^y b(s)\,ds\,dy\right]+e^{-X}\left(f(t,1)-f(t,0)\right),$$

we have an expression of the boundary layer term that controls the topography effect at the order 2.

In the interior of the domain, the difference between the two solutions, ϕ_{2int}, given by $\phi_{2int}(t,x,X)=\widetilde{\psi}_2^{int}(t,x,X)-\psi_2^{int}(t,x,X)$, is periodic in X. We define another function on $[0,+\infty[$, denoted by $\boldsymbol{G}$, periodic, with period P, solution of the following system:

$$\begin{aligned}
&\partial_X^2\boldsymbol{G}(X)+\partial_X\boldsymbol{G}(X)=b(X),\\
&\boldsymbol{G}(0)=\boldsymbol{G}(P)=0.
\end{aligned}$$

Then we can write ϕ_{2int} as:

$$\phi_{2int}(t,x,X)=\partial_x\psi_0^{int}(t,x)\boldsymbol{G}(X)+E(t,x)=-f(t,x)\boldsymbol{G}(X)+E(t,x),$$

$$\text{with}\begin{cases}\partial_xE(t,x)=\overline{\boldsymbol{G}(X)}^X\partial_xf(t,x)-\overline{b(X)\partial_X\boldsymbol{G}(X)}^X f(t,x),\\ E(t,1)=f(t,1)\boldsymbol{G}\left(\frac{1}{\varepsilon}\right).\end{cases}$$

Thus we can compute the term due to the topography that should be added to modify the solution in the interior of the domain at the order 2.

In the same way, we could express the topography terms for the upper orders but the results proposed here already give the first three orders of the approximation.

In order to prove the accuracy of this method, we performed the computations and we give the mains results in the following part.

4.5. Numerical experiments on the topography terms.

In this section, we compare the solution of the equation to the topography term to the solution of the equation for a flat bottom to which we add the topography terms we found in the previous section.

4.5.1. Computing details.

We consider the evolution equation but we choose f as a function of x only. We compute the solution with a flat bottom ψ and the solution with a varying bottom $\widetilde{\psi}$ using finite differences at the second order. As the stability condition (that reads $dt \leq 2dx^2\varepsilon^2/(4\varepsilon^2+dx^2)$) is strong, the computations are time expensive. We stop when the solution is stationary. Then we compare the function $\widetilde{\psi}$ and the sum of ψ and the topography terms ϕ_k (we only compute these terms for the final time).
For the topography terms, we use a LU decomposition to get the function $\boldsymbol{G}$, with derivatives at the second order. We take a constant space step as $\boldsymbol{G}$ is periodic (the values of $\boldsymbol{G}$ have been compared with its theoretical expression for various bottoms). We also need the expression of $\widetilde{\psi}_1^{cor}$: here we use the handwritten expression but in the following we will get it thanks to ψ_1^{cor} and ϕ_1. At last, we get the values of E with finite differences at the first order.

4.5.2. Numerical results.

In the following figures, we present the results for the bottom given by $1/2\cos(\pi X/10)+1/2\cos(\pi X/5)$ (we studied two other choices and the conclusions are the same). We have plotted, for $f(t,x)=x$, the differences between:

- ψ and $\widetilde{\psi}$,
- $\psi+\varepsilon\phi_1$ and $\widetilde{\psi}$,
- and between $\psi+\varepsilon\phi_1+\varepsilon^2\phi_{2CL}+\varepsilon^2\phi_{2int}=\psi+\varepsilon\phi_1+\varepsilon^2\phi_2$ and $\widetilde{\psi}$.

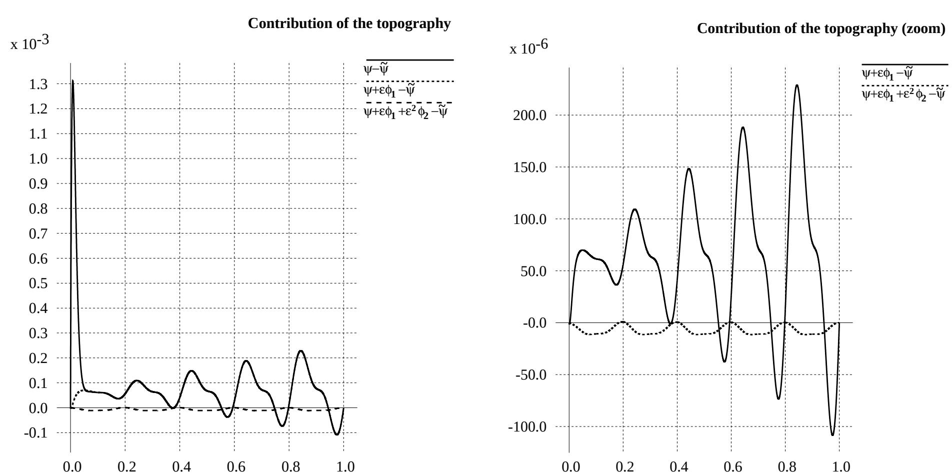

FIG. 4.1. *For $\varepsilon=0.01$, $dx=10^{-4}$, $dt=10^{-9}$ and $b(X)=1/2\cos(\pi X/10)+1/2\cos(\pi X/5)$: difference between $\widetilde{\psi}$ and the function ψ to which we added the topography terms.*

We remark that the difference between the solution of the equation with the topography and the solution for a flat bottom to which we added the topography terms is of the predicted order in ε.

Remark 4.3. *The curve of the difference $\psi - \widetilde{\psi}$ represents the effect of the topography. We have a logical oscillation similar to the bottom. However we can note a large influence in the boundary layer.*

4.6. Complete method.

The ultimate step is to replace the computation of the function ψ with finite differences by its approximation given thanks to the first orders of the asymptotic development. The results are plotted on Figure 4.2 (the bottom is still equal to $1/2\cos(\pi X/10) + 1/2\cos(\pi X/5)$).

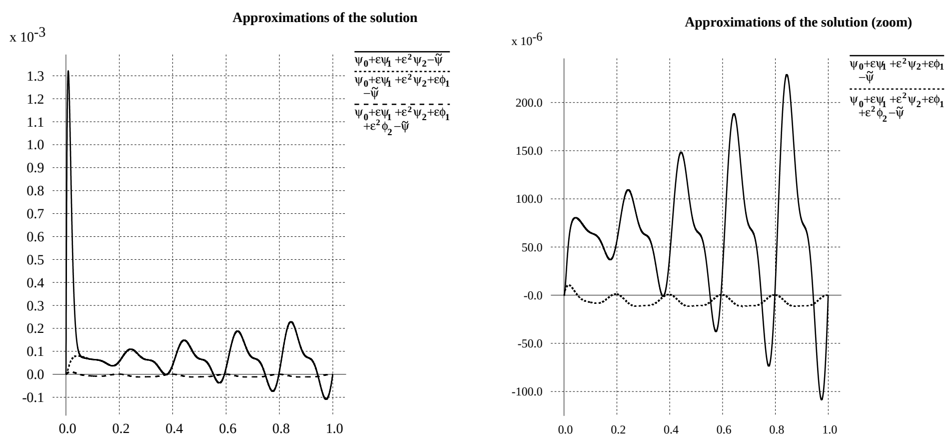

Fig. 4.2. *For $\varepsilon = 0.01$, $dx = 10^{-4}$, $dt = 10^{-4}$ and $b(X) = 1/2\cos(\pi X/10) + 1/2\cos(\pi X/5)$: difference between the function $\widetilde{\psi}$ and the asymptotic approximation of ψ to which we added the topography terms.*

We see that we are able to get an approximation of the oscillating solution at the order ε or ε^2.

Then we have an efficient method to compute the solution for a flat bottom. We can modify this function to get the solution of Equation (4.1) for any bottom. This program is rapid and can give an approximation of the solution at a determined order.

5. Conclusion

In this paper, we studied the Quasi-Geostrophic equation from two different points of view: first, with an oscillating topography, we introduced a new variable and a corrector term to take into account the quick scale and the boundary layer respectively. The first orders of the asymptotic development give us systems that have been implemented to get approximations of the solution. This method is efficient as it gives results comparable to those obtained with usual technics but it is less expensive in time and computations.

In the second part, we studied the role of the topography: we manage to find the terms that must be added to the solution with a flat bottom to take into account the topography. So we can solve the Quasi-Geostrophic equation without the topography term, and then add the contribution of the topography.

Acknowledgement. The authors would like to thank D. Bresch for fruitful discussions and commentaries around this work.

REFERENCES

[1] P. Ailliot, E. Frénod and V. Monbet, *Long term object drift forecast in the ocean with tide and wind*, Multiscale Modelling and Simulation, 5, 514–531, 2006.
[2] K. Bryan, *A Numerical Investigation of a Nonlinear Model of a Wind-Driven Ocean*, J. Atmos. Sci., 20, 594–606, 1963.
[3] D. Bresch and D. Gérard-Varet, *Roughness-Induced Effects on the Quasi-Geostrophic Model*, Comm. Math. Phys., 253, 81–119, 2005.
[4] J. G. Charney, *On the scale of atmospheric motions*, Geofys. Publ. (Oslo), 17, 17 pp., 1948.
[5] J. G. Charney, R. Fjörtoft and J. von Neuman, *Numerical integration of the barotropic vorticity equation*, Tellus, 2, 237–254, 1950.
[6] D. Gérard-Varet, *Highly rotating fluids in rough domains*, J. Math. Pures Appl., 82, 1453–1498, 2003.
[7] W. R. Holland, *The Role of Mesoscale Eddies in the General Circulation of the OceanNumerical Experiments Using a Wind-Driven Quasi-Geostrophic Model*, J. Phys. Oceanogr., 8, 363–392, 1978.
[8] C. Kazantsev, E. Kazantsev and E. Blayo, *Compact finite difference schemes for shallow-water ocean model*, Technical Report 4913, INRIA, 2003.
[9] R. Klein, E.Mikusky and A. Owinoh, *Multiple Scales Asymptotics for Atmospheric Flows*, in Proceedings of the 4th European Conference on Mathematics, Stockholm, Sweden, 2004.
[10] C. Lucas, *Effets de petites échelles, du tenseur des contraintes, des conditions au fond et à la surface sur les équations de Saint-Venant.* PhD, Grenoble, 2007.